

# Construction of dynamical invariants for the time-dependent harmonic oscillator with a time-dependent driven force

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## Abstract

We construct the linear and quadratic polynomial dynamical invariants for the classical and quantum time-dependent harmonic oscillator driven by a time-dependent force. To obtain them, we use exclusively the associated equations of motion for the system. We also find an algebraic relationship between the linear and quadratic invariants at the classical and quantum level.

## 1 Introduction

The driven time-dependent harmonic oscillator (driven TDHO) is one of the most useful models in modern and classical physics. As a simple model, it has several theoretical applications, *e.g.* the study of time evolution of quantum systems and its correlation with classical mechanics [1]. Also, it can be used to study the optimization of processes at the microscopic scale [2]. Moreover,

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it has experimental applications in areas such as molecular physics [3], ion traps [4, 5], quantum computation [6, 7], and accelerator physics [8].

Time dependent systems are not stationary, so the hamiltonian function is not a conserved quantity, thus conventional approaches to deal with time-independent systems cannot be used. Despite its explicit time-dependence, the dynamical invariant approach allows an appropriate analysis of the time-evolution of these systems in several situations, offering the possibility to find solutions of the classical and quantum dynamical equations. This analysis can be made in the cases in which the study of the conserved quantities associated to the symmetries of these systems can be implemented. In other words, in both classical and quantum cases, complete integration can be performed by finding its associated algebraic structure. That structure can be constructed with the operators that belong to conserved quantities which can be related to symmetries. The conserved quantities can be derived directly from the differential equations, as shown in [9], or by using variational approaches [10, 11].

The principal characteristic of the algebras associated with quadratic systems, like the ones managed in this work, is the existence of linear and quadratic operators associated to the dynamical invariants of the system. In general, there are two linear invariants for each degree of freedom. Using these operators, it can be generated a simple spectrum which, despite not being associated with the energy, allows us to find the solutions of the equations of motion [12].

Generally speaking, depending of the nature of the system, for each case, classical or quantum, we can use several methods to construct or find those dynamical invariants. Besides of the method proposed by the authors in [9], alternative procedures to obtain the dynamical invariants can be used depending on the system that is dealt with. For the quantum case we have Manko's *et al.* [4, 13] and the Lewis' procedures [12], and for the classical case we can find Lutzky's and Hojman's ones [10, 11].

In this work, we provide a continuation of the work [9], showing how to construct the dynamical invariants for the time dependent harmonic oscillator driven by a time dependent force, and deduce the relations between the linear and quadratic dynamical invariants for the classical and quantum problem. In sec. 2 we perform the derivation of linear and quadratic invariants from the classical equation of motion, and show that the quadratic one can be put in the form of product of two linear operators. The same is

done in sec. 3 for the quantum case. In sec. 4 we discuss the relationship between both linear and second-order invariants, in classical and quantum case. This analysis agrees with Manko's and Dodonov's idea [13] of the fundamental properties of the first-order integrals of motion. In sec. 5 we highlight our main results.

## 2 The classical driven TDHO

For the start point, let us take the second-order ODE that describes a one-dimensional classical time-dependent harmonic oscillator with a driven force,

$$\frac{d^2 q}{dt^2} + \omega^2(t) q = F(t), \quad (1)$$

in which the physical parameters as mass and strength of the driving term are normalized and absorbed by the variables. The frequency and the force term are considered to be analytic everywhere in  $t$ .

We may decompose (1) into the following first-order ODEs

$$p = \frac{dq}{dt}, \quad (2a)$$

and

$$\frac{dp}{dt} = F(t) - \omega^2(t) q. \quad (2b)$$

These may be recognized as the canonical equations of the driven TDHO. The most general solution of (2) requires two initial data, which may be the values of  $q$  and  $p$  at an instant of time  $t = t_0$ . The integrability of these equations are then determined by the knowledge of two linearly independent dynamical invariants of the system. The canonical equations themselves has all the information necessary to find these quantities.

### 2.1 Linear invariants

Let us proceed with the same procedure done in [9], defining two arbitrary complex functions  $\alpha(t)$  and  $\beta(t)$ . Multiplying (2a) by  $\alpha$  and (2b) by  $\beta$ , then building the linear combination, yields

$$\beta \frac{dp}{dt} + \alpha \frac{dq}{dt} = \alpha p + \beta (F - \omega^2 q).$$

Isolating the total time derivative results in the expression

$$\frac{d}{dt}(\beta p + \alpha q) = \left(\alpha + \frac{d\beta}{dt}\right)p + \left(\frac{d\alpha}{dt} - \omega^2\beta\right)q + \beta F. \quad (3)$$

The force term in (3) becomes a problem, since we seek for conditions over  $\alpha$  and  $\beta$  so the l.h.s. can be made zero. We may fix this defining a function  $\mathcal{F}(\beta, t)$  such that

$$\mathcal{F}(\beta, t) \equiv \int_{t_0}^t \beta(\tau) F(\tau) d\tau, \quad \beta(t_0) = 0. \quad (4)$$

In this case we have the identity

$$\beta F = \frac{d\mathcal{F}}{dt}. \quad (5)$$

With (5), we may express (3) in the form

$$\frac{d}{dt}(\beta p + \alpha q - \mathcal{F}) = \left(\alpha + \frac{d\beta}{dt}\right)p + \left(\frac{d\alpha}{dt} - \omega^2\beta\right)q. \quad (6)$$

If the parameters  $\alpha$  and  $\beta$  satisfy the ODEs

$$\alpha + \frac{d\beta}{dt} = 0, \quad (7a)$$

$$\frac{d\alpha}{dt} - \omega^2\beta = 0, \quad (7b)$$

the polynomial

$$I_L = \beta p + \alpha q - \mathcal{F}(\beta, t) \quad (8)$$

is a linear dynamical invariant of the system (2). We see by (7) that  $\alpha$  and  $\beta$  are not independent functions, so we may write (8) depending only of  $\beta$ :

$$I_L = \beta p - \frac{d\beta}{dt}q - \mathcal{F}(\beta, t), \quad (9a)$$

where  $\beta$  obeys

$$\left(\frac{d^2}{dt^2} + \omega^2\right)\beta = 0. \quad (9b)$$

The most general solution of (9b) is a complex function  $\beta$ , so we actually

have two linearly independent invariants

$$I_L = \beta p - \frac{d\beta}{dt} q - \mathcal{F}(\beta, t), \quad (10a)$$

$$I_L^* = \beta^* p - \frac{d\beta^*}{dt} q - \mathcal{F}(\beta^*, t), \quad (10b)$$

where  $\beta$  and  $\beta^*$  obey the set of ODEs

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \begin{pmatrix} \beta \\ \beta^* \end{pmatrix} = 0. \quad (11)$$

Eqs. (11) are the same expected for the parameter  $\beta$  in the case of a simple time-dependent harmonic oscillator [9].

## 2.2 Quadratic invariants

We may also build quadratic invariants from the equations of motion (2). Without the driving force, it would be sufficient to build linear combinations of products of the equations of motion. However this is not the case when the driving force is in place. Let us observe the following products between (2a) and (2b):

$$\frac{d}{dt} \left( \frac{q^2}{2} \right) = qp, \quad (12a)$$

$$\frac{d}{dt} (qp) = p^2 - \omega^2 q^2 + qF, \quad (12b)$$

$$\frac{d}{dt} \left( \frac{p^2}{2} \right) = (F - \omega^2 q) \frac{dq}{dt}. \quad (12c)$$

The r.h.s. of these equations fail to be purely quadratic forms in the variables  $(q, p)$ , because of the presence of the driving force. This situation is corrected with the use of the equations of motion themselves:

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = (F - \omega^2 q).$$

Now we take a set of time-dependent functions  $c_i = (c_1, c_2, c_3, c_4, c_5)$ , and build the linear combination

$$\begin{aligned} c_1 \frac{d}{dt} \left( \frac{q^2}{2} \right) + c_2 \frac{d}{dt} (qp) + c_3 \frac{d}{dt} \left( \frac{p^2}{2} \right) + c_4 \frac{dq}{dt} + c_5 \frac{dp}{dt} = \\ = c_1 qp + c_2 (p^2 - \omega^2 q^2 + qF) + c_3 (F - \omega^2 q^2) + c_4 p + c_5 (F - \omega^2 q). \end{aligned}$$

Collecting the total time derivatives, it yields the expression

$$\begin{aligned}
\frac{d}{dt} \left( c_1 \frac{q^2}{2} + c_2 qp + c_3 \frac{p^2}{2} + c_4 q + c_5 p \right) &= \\
&= \left( c_2 + \frac{1}{2} \frac{dc_3}{dt} \right) p^2 + \left( \frac{1}{2} \frac{dc_1}{dt} - c_2 \omega^2 \right) q^2 + \left( c_1 + \frac{dc_2}{dt} - c_3 \omega^2 \right) qp \\
&\quad + \left( c_2 F + \frac{dc_4}{dt} - c_5 \omega^2 \right) q + \left( c_3 F + c_4 + \frac{dc_5}{dt} \right) p + c_5 F. \tag{13}
\end{aligned}$$

Now we use the identity

$$c_5 F = \frac{d\mathcal{F}(c_5, t)}{dt} = \frac{d}{dt} \left[ \int_{t_0}^t c_5(\tau) F(\tau) d\tau \right], \quad c_5(t_0) = 0, \tag{14}$$

and in this case,

$$\begin{aligned}
\frac{d}{dt} \left[ c_1 \frac{q^2}{2} + c_2 qp + c_3 \frac{p^2}{2} + c_4 q + c_5 p - \mathcal{F}(c_5, t) \right] &= \\
&= \left( c_2 + \frac{1}{2} \frac{dc_3}{dt} \right) p^2 + \left( \frac{1}{2} \frac{dc_1}{dt} - c_2 \omega^2 \right) q^2 + \left( c_1 + \frac{dc_2}{dt} - c_3 \omega^2 \right) qp \\
&\quad + \left( c_2 F + \frac{dc_4}{dt} - c_5 \omega^2 \right) q + \left( c_3 F + c_4 + \frac{dc_5}{dt} \right) p. \tag{15}
\end{aligned}$$

Therefore the second-order polynomial

$$I = \frac{c_1}{2} q^2 + c_2 qp + \frac{c_3}{2} p^2 + c_4 q + c_5 p - \mathcal{F}(c_5, t) \tag{16}$$

is a dynamical invariant if the equations

$$c_2 + \frac{1}{2} \frac{dc_3}{dt} = 0, \tag{17a}$$

$$\frac{1}{2} \frac{dc_1}{dt} - c_2 \omega^2 = 0, \tag{17b}$$

$$c_1 + \frac{dc_2}{dt} - c_3 \omega^2 = 0, \tag{17c}$$

$$c_2 F + \frac{dc_4}{dt} - c_5 \omega^2 = 0, \tag{17d}$$

$$c_3 F + c_4 + \frac{dc_5}{dt} = 0 \tag{17e}$$

are satisfied.

We notice that (16) can be rewritten to depend only of the functions  $c_3$  and  $c_5$ . Let us rename them as  $\gamma$  and  $\sigma$  respectively. This can be done with the set

(17), and results in

$$I_Q = \left( \frac{1}{2} \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma \right) \frac{q^2}{2} - \frac{1}{2} \frac{d\gamma}{dt} qp + \gamma \frac{p^2}{2} - \left( \frac{d\sigma}{dt} + \gamma F \right) q + \sigma p - \mathcal{F}(\sigma, t). \quad (18)$$

The ODEs for  $\gamma$  and  $\sigma$  follow:

$$\frac{1}{2} \frac{d^3 \gamma}{dt^3} + 2\omega^2 \frac{d\gamma}{dt} + \frac{d\omega^2}{dt} \gamma = 0, \quad (19a)$$

$$\frac{d^2 \sigma}{dt^2} + \omega^2 \sigma = -\gamma \frac{dF}{dt} - \frac{3}{2} \frac{d\gamma}{dt} F. \quad (19b)$$

More than that, (19a) can be integrated to give

$$\left( \frac{1}{2} \frac{d^2}{dt^2} + \omega^2 \right) \gamma = \frac{W^2}{\gamma} + \frac{1}{4\gamma} \left( \frac{d\gamma}{dt} \right)^2, \quad (20)$$

in which  $W^2$  is the integration constant.

Supposing a given solution of (20), a function  $\sigma$  can be found as a solution of (19b). In this case, because of the force term, two independent parameters are required to the construction of the second-order invariant (18). Using (20) in (18) we have

$$I_Q = \frac{1}{2\gamma} \left[ W^2 q^2 + \left( \frac{1}{2} \frac{d\gamma}{dt} q - \gamma p \right)^2 \right] - \left( \frac{d\sigma}{dt} + \gamma F \right) q + \sigma p - \mathcal{F}(\sigma, t). \quad (21)$$

The invariant (21) is found by K. Takayama in [8] using the dynamical algebra of the related hamiltonian function, and it is also used to model charged particles in betatron accelerators.

Let us suppose that  $\gamma$  is a positive real quantity, i.e.  $\gamma \equiv \rho^2$ . In this case, (20) becomes

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \rho = \frac{W^2}{\rho^3}, \quad (22)$$

which is an Ermakov equation for the parameter  $\rho$ , and

$$I_Q = \frac{1}{2} \left[ \frac{q^2}{\rho^2} + \left( \frac{d\rho}{dt} q - \rho p \right)^2 \right] - \left( \frac{d\sigma}{dt} + \rho^2 F \right) q + \sigma p - \mathcal{F}(\sigma, t) \quad (23)$$

can be seen as an Ermakov-like invariant of the driven TDHO.

### 3 The quantum driven TDHO

Let us now deal with the quantum analogue of the problem dealt in sec. 2. The quantum oscillator is described by the set of variables  $(\hat{q}, \hat{p})$ , which are now self-adjoint operators acting on elements of a Hilbert space. The equations of motion in this case become Heisenberg equations for these operators, and their functional form is identical compared to the classical system, i.e.,

$$\hat{p} = \frac{d\hat{q}}{dt}, \quad \frac{d\hat{p}}{dt} = F(t) - \omega^2(t) \hat{q}. \quad (24)$$

The operators  $\hat{q}$  and  $\hat{p}$  do not commute, i.e.  $[\hat{q}, \hat{p}] \equiv \hat{q}\hat{p} - \hat{p}\hat{q} \neq 0$ , therefore the ordering problem between these operators has to be taken into account. The force term is understood to be the unity operator multiplied by an analytic time dependent function  $F(t)$ .

#### 3.1 Quantum invariants

Since the ordering problem does not affect linear combinations of the eqs. (24), the same construction of sec. 2.1 can be made. The result is simply that the operators

$$\hat{I}_L = \beta \hat{p} - \frac{d\beta}{dt} \hat{q} - \mathcal{F}(\beta, t), \quad (25a)$$

$$\hat{I}_L^\dagger = \beta^* \hat{p} - \frac{d\beta^*}{dt} \hat{q} - \mathcal{F}(\beta^*, t), \quad (25b)$$

in which

$$\mathcal{F}(\beta, t) \equiv \int_{t_0}^t \beta(\tau) F(\tau) d\tau, \quad \beta(t_0) = 0, \quad (26)$$

are linear invariants of (24) if  $\beta$  and  $\beta^*$  obey the equations

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \begin{pmatrix} \beta \\ \beta^* \end{pmatrix} = 0, \quad (27)$$

which are the same eqs. (11).

The construction of the quadratic dynamical invariants, on the other hand, needs special attention to the ordering problem. Taking (24), we may build



the products

$$\frac{d\hat{q}^2}{dt} = \{\hat{q}, \hat{p}\}, \quad (28a)$$

$$\frac{d}{dt} \{\hat{q}, \hat{p}\} = 2\hat{p}^2 + 2F\hat{q} - 2\omega^2\hat{q}^2, \quad (28b)$$

$$\frac{d\hat{p}^2}{dt} = 2F\hat{p} - \omega^2 \{\hat{q}, \hat{p}\}, \quad (28c)$$

where  $\{\hat{q}, \hat{p}\} \equiv \hat{q}\hat{p} + \hat{p}\hat{q}$  is the anti-commutator. Once again the presence of the force term requires the use of the Heisenberg equations themselves:

$$\frac{d\hat{q}}{dt} = \hat{p}, \quad \frac{d\hat{p}}{dt} = F - \omega^2\hat{q}.$$

We define a set of time-dependent functions  $c_i = (c_1, c_2, c_3, c_4, c_5)$ , and build the linear combination between these equations. Taking the total time-derivative gives the expression

$$\begin{aligned} \frac{d}{dt} \left[ c_1 \frac{\hat{q}^2}{2} + \frac{1}{2} c_2 \{\hat{q}, \hat{p}\} + c_3 \frac{\hat{p}^2}{2} + c_4 \hat{q} + c_5 \hat{p} - \mathcal{F}(c_5, t) \right] = \\ = \left( c_2 + \frac{1}{2} \frac{dc_3}{dt} \right) \hat{p}^2 + \left( \frac{1}{2} \frac{dc_1}{dt} - c_2 \omega^2 \right) \hat{q}^2 + \frac{1}{2} \left( c_1 + \frac{dc_2}{dt} - c_3 \omega^2 \right) \{\hat{q}, \hat{p}\} \\ + \left( c_2 F + \frac{dc_4}{dt} - c_5 \omega^2 \right) \hat{q} + \left( c_3 F + c_4 + \frac{dc_5}{dt} \right) \hat{p}. \end{aligned} \quad (29)$$

Using (26), we reach the quantum second-order polynomial

$$\hat{I}_Q = c_1 \frac{\hat{q}^2}{2} + \frac{1}{2} c_2 \{\hat{q}, \hat{p}\} + c_3 \frac{\hat{p}^2}{2} + c_4 \hat{q} + c_5 \hat{p} - \mathcal{F}(c_5, t), \quad (30)$$

which is a dynamical invariant if the set

$$c_2 + \frac{1}{2} \frac{dc_3}{dt} = 0, \quad (31a)$$

$$\frac{1}{2} \frac{dc_1}{dt} - c_2 \omega^2 = 0, \quad (31b)$$

$$c_1 + \frac{dc_2}{dt} - c_3 \omega^2 = 0, \quad (31c)$$

$$c_2 F + \frac{dc_4}{dt} - c_5 \omega^2 = 0, \quad (31d)$$

$$c_3 F + c_4 + \frac{dc_5}{dt} = 0 \quad (31e)$$

is satisfied. These are the same equations (18).

Again, using (31) it is possible to express (30) in terms of  $c_3 \equiv \gamma$  and  $c_5 \equiv \sigma$ :

$$\hat{I}_Q = \left( \frac{1}{2} \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma \right) \frac{\hat{q}^2}{2} - \frac{1}{4} \frac{d\gamma}{dt} \{ \hat{q}, \hat{p} \} + \gamma \frac{\hat{p}^2}{2} - \left( \frac{d\sigma}{dt} + \gamma F \right) \hat{q} + \sigma \hat{p} - \mathcal{F}(\sigma, t). \quad (32)$$

The equations for  $\gamma$  and  $\sigma$  are also the same of the classical system:

$$\frac{1}{2} \frac{d^3 \gamma}{dt^3} + 2\omega^2 \frac{d\gamma}{dt} + \frac{d\omega^2}{dt} \gamma = 0, \quad (33a)$$

$$\frac{d^2 \sigma}{dt^2} + \omega^2 \sigma = -\gamma \frac{dF}{dt} - \frac{3}{2} \frac{d\gamma}{dt} F. \quad (33b)$$

The  $\gamma$  equation can be again integrated to give the second-order ODE (20). Eqs. (21), (22), and (23) also follow straightforwardly.

## 4 Algebra of the dynamical invariants

### 4.1 Classical case

In sec. 2 we see that the linear invariants for the driven TDHO are given by

$$I_L = \beta p - \frac{d\beta}{dt} q - \mathcal{F}(\beta, t), \quad (34a)$$

$$I_L^* = \beta^* p - \frac{d\beta^*}{dt} q - \mathcal{F}(\beta^*, t), \quad (34b)$$

provided that  $\beta$  and  $\beta^*$  obey

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \begin{pmatrix} \beta \\ \beta^* \end{pmatrix} = 0, \quad (35)$$

and

$$\mathcal{F}(f, t) = \int_{t_0}^t f(\tau) F(\tau) d\tau, \quad f(t_0) = 0, \quad (36)$$

for a generic function  $f(t)$ .

On the other hand, the quadratic form

$$I_Q = \left( \frac{1}{2} \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma \right) \frac{q^2}{2} - \frac{1}{2} \frac{d\gamma}{dt} qp + \gamma \frac{p^2}{2} - \left( \frac{d\sigma}{dt} + \gamma F \right) q + \sigma p - \mathcal{F}(\sigma, t) \quad (37)$$

is a second-order invariant of the driven TDHO if  $\gamma$  and  $\sigma$  obey the set

$$\frac{1}{2} \frac{d^3 \gamma}{dt^3} + 2\omega^2 \frac{d\gamma}{dt} + \frac{d\omega^2}{dt} \gamma = 0, \quad (38a)$$

$$\frac{d^2 \sigma}{dt^2} + \omega^2 \sigma = -\gamma \frac{dF}{dt} - \frac{3}{2} \frac{d\gamma}{dt} F. \quad (38b)$$

We now ask the question if the quadratic invariant can be related with the linear ones.

Let us consider the product

$$\begin{aligned} I_L^* I_L = & \frac{d\beta^*}{dt} \frac{d\beta}{dt} q^2 - \frac{d}{dt} (\beta^* \beta) qp + (\beta^* \beta) p^2 + \left[ \frac{d\beta^*}{dt} \mathcal{F}(\beta, t) + \frac{d\beta}{dt} \mathcal{F}(\beta^*, t) \right] q \\ & - [\beta^* \mathcal{F}(\beta, t) + \beta \mathcal{F}(\beta^*, t)] p + |\mathcal{F}(\beta, t)|^2. \end{aligned} \quad (39)$$

Any product of invariants is also an invariant. In the case of (39), it becomes a second-order real invariant in the variables  $p$  and  $q$ . Observing (37), both quadratic forms are equivalent if

$$\gamma = 2\beta^* \beta, \quad (40a)$$

$$\frac{1}{2} \left( \frac{1}{2} \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma \right) = \frac{d\beta^*}{dt} \frac{d\beta}{dt}, \quad (40b)$$

$$\frac{d\sigma}{dt} + \gamma F = -\frac{d\beta^*}{dt} \mathcal{F}(\beta, t) - \frac{d\beta}{dt} \mathcal{F}(\beta^*, t), \quad (40c)$$

$$\sigma = -\beta^* \mathcal{F}(\beta, t) - \beta \mathcal{F}(\beta^*, t), \quad (40d)$$

$$\mathcal{F}(\sigma, t) = -|\mathcal{F}(\beta, t)|^2. \quad (40e)$$

are satisfied.

Let us observe the term on the l.h.s. of (40b). Using (40a) we have

$$\frac{1}{2} \left( \frac{1}{2} \frac{d^2}{dt^2} + \omega^2 \right) \gamma = \frac{1}{2} \left( \frac{d^2 \beta^*}{dt^2} + \omega^2 \beta^* \right) \beta + \frac{1}{2} \left( \frac{d^2 \beta}{dt^2} + \omega^2 \beta \right) \beta^* + \frac{d\beta^*}{dt} \frac{d\beta}{dt}.$$

Since  $\beta$  and  $\beta^*$  obey (35), (40b) is identically satisfied.

Now we take (40d) and substitute  $\sigma$  in the l.h.s. of eq. (40c). The result is

$$\frac{d\sigma}{dt} + \gamma F = -\frac{d\beta^*}{dt} \mathcal{F}(\beta, t) - \frac{d\beta}{dt} \mathcal{F}(\beta^*, t),$$

which is exactly (40c). On the other hand, (40d) implies

$$\mathcal{F}(\sigma, t) = -|\mathcal{F}(\beta, t)|^2.$$

Then, (40e) is also identically satisfied.

We still have to show that (40a) and (40d) are solutions of (38). In the case of 38a, it is more convenient to work with the equivalent equation

$$\frac{d}{dt} \left[ \frac{1}{2} \gamma \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma^2 - \frac{1}{4} \left( \frac{d\gamma}{dt} \right)^2 \right] = 0. \quad (41)$$

Inside the brackets, using (35) and (40a), we have

$$\frac{1}{2} \gamma \frac{d^2 \gamma}{dt^2} + \omega^2 \gamma^2 - \frac{1}{4} \left( \frac{d\gamma}{dt} \right)^2 = - \left( \frac{d\beta^*}{dt} \beta - \beta^* \frac{d\beta}{dt} \right)^2.$$

The quantity

$$W(\beta^*, \beta) = \frac{d\beta^*}{dt} \beta - \beta^* \frac{d\beta}{dt} \quad (42)$$

is the Wronskian of the functions  $\beta^*$  and  $\beta$ . From (35), it is straightforward to show that the Wronskian (42) is a constant of motion. Therefore, if (35) is true, (41) is identically satisfied. In fact the integration constant in (20) is precisely the square of the Wronskian (42).

Let us see what happens with (38b). The l.h.s. yields

$$\begin{aligned} \left( \frac{d^2}{dt^2} + \omega^2 \right) \sigma &= -\mathcal{F}(\beta, t) \left( \frac{d^2}{dt^2} + \omega^2 \right) \beta^* - \mathcal{F}(\beta^*, t) \left( \frac{d^2}{dt^2} + \omega^2 \right) \beta \\ &\quad - \frac{d}{dt} (\beta^* \beta) F - 2 \frac{d}{dt} (\beta^* \beta F). \end{aligned}$$

Again using (35),

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \sigma = -\gamma \frac{dF}{dt} - \frac{3}{2} \frac{d\gamma}{dt} F,$$

where (40a) is used. This reproduces eq. (38b), as required.

## 4.2 Quantum case

Due to the Heisenberg algebra of the quantum driven TDHO, it is necessary to consider symmetric and antisymmetric products of the linear invariants (25). The antisymmetric product is given by

$$\hat{I}_A = \frac{1}{2} [\hat{I}_L^\dagger, \hat{I}_L] = -\frac{1}{2} W(\beta^*, \beta) [\hat{q}, \hat{p}],$$

which is not a dynamical invariant, since it is a pure constant term. On the other hand, the symmetric product

$$\begin{aligned}\hat{I}_S = \frac{1}{2} \left\{ \hat{I}_L^\dagger, \hat{I}_L \right\} &= \beta^* \beta \hat{p}^2 - \frac{1}{2} \frac{d}{dt} (\beta^* \beta) \{ \hat{q}, \hat{p} \} + \frac{d\beta^*}{dt} \frac{d\beta}{dt} \hat{q}^2 \\ &\quad - [\beta^* \mathcal{F}(\beta, t) + \mathcal{F}(\beta^*, t) \beta] \hat{p} + \left[ \frac{d\beta^*}{dt} \mathcal{F}(\beta, t) + \mathcal{F}(\beta^*, t) \frac{d\beta}{dt} \right] \hat{q} + |\mathcal{F}(\beta, t)|^2\end{aligned}\tag{43}$$

is a true quadratic invariant.

Let us seek for the conditions that allow (43) to be equal to (32). We find that the following set of equations:

$$\gamma = 2\beta^* \beta, \tag{44a}$$

$$\sigma = -\beta^* \mathcal{F}(\beta, t) - \mathcal{F}(\beta^*, t) \beta, \tag{44b}$$

together with

$$\frac{1}{2} \frac{d^2 \gamma}{dt^2} + \gamma \omega^2 = 2 \frac{d\beta^*}{dt} \frac{d\beta}{dt}, \tag{45a}$$

$$\frac{d\sigma}{dt} + \gamma F = -\frac{d\beta^*}{dt} \mathcal{F}(\beta, t) - \mathcal{F}(\beta^*, t) \frac{d\beta}{dt}, \tag{45b}$$

$$\mathcal{F}(c_5, t) = -|\mathcal{F}(\beta, t)|^2 \tag{45c}$$

are required. As for the classical case, eqs. (44) implies (45). Also, (44) are solutions of (33) if  $\beta$  and  $\beta^*$  are solutions of (27).

## 5 Final remarks

In this paper we analysed the problem of construction of linear and quadratic dynamical invariants for the classical and quantum driven time-dependent harmonic oscillator. The procedure, applied in [9] for the case of the unforced harmonic oscillator, requires nothing more than the first-order equations of motion (or the Heisenberg equations in the quantum case). As for the unforced case, the linear dynamical invariants (10) ((25) in the quantum case) of the driven TDHO are built from linear combinations of the equations of motion (2) ((24)), provided the complex parameter  $\beta$  and its conjugated  $\beta^*$  obey the ODEs (11). We observed that the presence of the driven force does not modify the equations for  $\beta$ , although the functional form of the linear invariants is modified by the introduction of the function  $\mathcal{F}(\beta, t)$ .

In the case of quadratic invariants (18) ((32)), they can be constructed as linear combinations of quadratic products of the first-order equations of motion, but because of the existence of the driven force, they also need the contribution of simple linear combinations of the first-order equations themselves. Two independent parameters  $\gamma$  and  $\sigma$  are required to obey eqs. (19). While  $\sigma$  does not exist in the unforced problem, the equation for  $\gamma$  is exactly the one found in [9]. The existence of the driven force yields, in this case, a second-order invariant which is not a pure quadratic form, but depends linearly of the variables  $q$  and  $p$  (or the respective Hilbert space operators in the quantum case).

The most interesting result in this method is the fact that the linear invariants can be used as building blocks for the construction of second-order invariants. This is illustrated in [9] for the TDHO, and now in sec. 4 for the driven TDHO. The second-order invariant (37) ((32)) is achieved from products of the linear invariants. In the classical case, the necessary and sufficient conditions for the product  $I_L^* I_L$  to be equal to the quadratic invariant (37) is that  $I_L$  and  $I_L^*$  must be linear dynamical invariants, and the parameters  $\gamma$  and  $\sigma$  must be related to  $\beta$  and  $\beta^*$  through eqs. (40a) and (40d). In the quantum case, one must consider the symmetric product (43) between  $\hat{I}_L$  and  $\hat{I}_L^\dagger$ , and the result is that the same theorem of the classical case is proved. This result agrees with Man'ko's and Dodonov's statement [13], that linear invariants are the fundamental quantities related to the quantization of dynamical systems.

The proposed method allows the study of the classical and quantum systems dynamics without using either hamiltonian or lagrangian formulations, since we need only the equations of motion to study the symmetries and construct the *algebrae* associated to the dynamics of the system. Following this idea, it is possible the study of systems in which the definition of such dynamical functions is not well defined; as far as we know, the study of the symmetries of any system in which the dynamical equations are linear. As an example, we mention systems with dissipative interactions up to second-order in the velocities. Those physical systems, highly studied in the analysis of more realistic systems, have a problematic definition of the hamiltonian or lagrangian functions and the construction of the quantization procedures.

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